Deduction of CANONICAL equation from a variational Principle →

Hamiltonian Principle is stated as

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt = 0$$

Hamiltonian $H = \sum_j p_j \dot{q}_j - L$

so
$$\delta I = \delta \int_{t_1}^{t_2} \left[\sum_j p_j \dot{q}_j - H(q_j, p_j, t)\right] dt = 0$$

Modified Hamiltonian's principle interms of parameter, α can be expressed

$$\delta I = d\alpha \frac{\partial I}{\partial \alpha} = d\alpha \frac{\partial}{\partial \alpha}\int \left[\sum_j p_j \dot{q}_j - H(q_j, p_j, t)\right] dt = 0$$

since end points time are same for every path, limits are independent of α Hence

$$\delta I = d\alpha \int_{t_1}^{t_2} \left[\sum_j \frac{\partial p_j}{\partial \alpha}\dot{q}_j + p_j \frac{\partial \dot{q}_j}{\partial \alpha} - \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial \alpha} - \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial \alpha} - \frac{\partial H}{\partial t}\frac{\partial t}{\partial \alpha}\right] dt = 0$$

time of travel along every path is same so $\partial t/\partial \alpha = 0$

$$\int_{t_1}^{t_2} p_j \frac{\partial \dot{q}_j}{\partial \alpha} dt = \int_{t_1}^{t_2} p_j \frac{d}{dt}\left(\frac{\partial q_j}{\partial \alpha}\right) dt = p_j \frac{\partial q_j}{\partial \alpha}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_j \frac{\partial q_j}{\partial \alpha} dt$$

$$= -\int_{t_1}^{t_2} \dot{p}_j \frac{\partial q_j}{\partial \alpha} dt$$

$\frac{\partial q_J}{\partial \alpha}$ = vanish because all paths have the same terminal

$$\delta I = d\alpha \int_{t_1}^{t_2} \sum_J \left(\frac{\partial p_J}{\partial \alpha} \dot{q}_J - \dot{p}_J \frac{\partial q_J}{\partial \alpha} - \frac{\partial H}{\partial q_J} \cdot \frac{\partial q_J}{\partial \alpha} - \frac{\partial H}{\partial p_J} \cdot \frac{\partial p_J}{\partial \alpha} \right) dt = 0$$

putting $\frac{\partial q_J}{\partial \alpha} d\alpha = \delta q_J$

$$\frac{\partial p_J}{\partial \alpha} d\alpha = \delta p_J$$

$$\delta I = \int_{t_1}^{t_2} \sum_J \left(\delta p_J \dot{q}_J - \dot{p}_J \delta q_J - \frac{\partial H}{\partial q_J} \delta q_J - \frac{\partial H}{\partial p_J} \delta p_J \right) dt$$

$$= \int_{t_1}^{t_2} \sum_J \left\{ \delta p_J \left(\dot{q}_J - \frac{\partial H}{\partial p_J} \right) - \delta q_J \left(\dot{p}_J + \frac{\partial H}{\partial q_J} \right) \right\} dt = 0$$

Since p_J and q_J are independent variables their variation δp_J and δq_J will also be independent of each other so above integral can vanish only if the coefficients separately vanish. i.e

$$\left.\begin{aligned} \dot{q}_J - \frac{\partial H}{\partial p_J} &= 0 \quad \text{or} \quad \dot{q}_J = \frac{\partial H}{\partial p_J} \\ \dot{p}_J + \frac{\partial H}{\partial q_J} &= 0 \quad \text{or} \quad \dot{p}_J = -\frac{\partial H}{\partial q_J} \end{aligned}\right\}$$

These are canonical eqn of motion.